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A kernel for automorphic L-functions

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Abstract

The full multiple Dirichlet series of an automorphic cusp form is defined, in classical language, as a Dirichlet series of several complex variables over all the Fourier coefficients of the cusp form. It is different from the L-function of Godement and Jacquet, which is defined as a Dirichlet series in one complex variable over a one-dimensional array of the Fourier coefficients. In $GL(2)$ and $GL(3)$, the two notions are simply related. In this paper, we construct a kernel function that gives the full multiple Dirichlet series of automorphic cusp forms on $GL(n,\mathbb{R})$. The kernel function is a new Poincaré series. Specifically, the inner product of a cusp form with this Poincaré series is the product of the full multiple Dirichlet series of the form times a function that is essentially the Mellin transform of Jacquet's Whittaker function. In the proof, the full multiple Dirichlet series is produced by applying the Lipschitz summation formula several times and by an integral which collapses the sum over $SL(n-1,\mathbb{Z})$ in the Fourier expansion of the cusp form.

Contents

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In order to approach the Ramanujan conjecture for $GL(2)$ and obtain a bound on the nth Fourier coefficient of a cusp form, Selberg [16] studied the following linear functional for fixed n:

$$\phi \to a_n,$$

where ϕ is a Maass cusp form for $GL(2)$ and a_n is its nth Fourier coefficient. Selberg noted that the linear functional can be realized as an inner product against a Poincaré series and obtained a bound on the coefficient of a Maass cusp form for $GL(2)$. Selberg's bound has since been improved though the Ramanujan conjecture for Maass cusp forms is still open. The Ramanujan conjecture for holomorphic cusp forms for congruence subgroups of $SL(2,\mathbb{Z})$ was established by Deligne.

As a means of studying the Lindelöf hypothesis, Anton Good [11,12] investigated the linear functional for fixed $s \in \mathbb{C}$, $\mathrm{Re}(s) \gg 1$ that sends a cusp form to its L-function:

$$\phi \to L_\phi(s).$$

He created a Poincaré series to serve as the kernel of integration and used it to calculate the second moment of holomorphic cusp forms of integral weight k for $SL(2,\mathbb{Z})$. In his method, a function resembling

$$D(w) = \int_0^\infty \left| L_\phi\left(\frac{k}{2} + it\right)\right|^2 g_k(t,w)\,dt,$$

is created and studied where $g_k(t,w) \sim t^{-w}$ as $t \to \infty$.

More recently, Adrian Diaconu, Dorian Goldfeld and Jeffrey Hoffstein [5] have suggested and explored the idea of associating to all classical moment problems a multiple Dirichlet series in the spirit of Good.

In this paper, we will construct a new Poincaré series which can be used as the kernel of integration to obtain the full multiple Dirichlet series of automorphic cusp forms on $GL(n,\mathbb{R})$. We will relate our Poincaré series to the classical one studied by Selberg and generalized by Bump, Friedberg and Goldfeld [4].

An application of our methods on $GL(2)$ can be found in [18] where we compute a mixed moment of the L-functions of holomorphic cusp forms on congruence subgroups. Another application of the linear functional that sends forms to their L-functions can be found in [10].

1. Introduction to $GL(n)$

The generalized upper half-plane of level $n \geqslant 2$ is defined as a homogeneous space

$$\mathcal{H}^n = GL(n,\mathbb{R})/\big(O(n,\mathbb{R})Z(n,\mathbb{R})\big).$$

The group $O(n,\mathbb{R})$ is the orthogonal group and $Z(n,\mathbb{R})$ is the center of $GL(n,\mathbb{R})$. The center, $Z(n,\mathbb{R})$, consists of all the non-zero scalar matrices in $GL(n,\mathbb{R})$. The space $\mathcal{H}^2$ gives the classical upper half of the plane of complex numbers.

From the Iwasawa decomposition theorem, every $z \in \mathcal{H}^n$ has the unique form $z = x(z)\cdot y(z)$ where $x(z)$ is an upper triangular matrix with 1s on the diagonal and $y(z)$ is a diagonal matrix with positive entries. For $z \in \mathcal{H}^n$ define the real coordinates $x_{i,j}$ for $1 \leqslant i < j \leqslant n$, and $y_k > 0$ for $1 \leqslant k \leqslant n-1$ as below:

$$z = x(z)y(z),$$

$$z = \begin{pmatrix} 1 & \dots & x_{i,j} \\ & 1 & \vdots \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \dots y_{n-1} & & & & \\ & y_1 \dots y_{n-2} & & & \\ & & \ddots & & \\ & & & y_1 & \\ & & & & 1 \end{pmatrix}.$$

The group $GL(n, \mathbb{R})$ acts on $\mathcal{H}^n$ by left matrix multiplication. The left-$GL(n, \mathbb{R})$ invariant measure on $\mathcal{H}^n$ is defined by $d^*z = d^*x \cdot d^*y$ where the measures d^*x and d^*y are given below as the wedge products of differential one-forms:

$$d^*x = \prod_{\substack{i,j=1 \\ i<j}}^{n} dx_{i,j} \quad \text{and} \quad d^*y = \prod_{i=1}^{n-1} y_i^{-i(n-i)-1}\, dy_i.$$

With the measure on $\mathcal{H}^n$, we may define the Hilbert space of square-integrable functions that are $SL(n, \mathbb{Z})$ invariant.

Definition 1. Let

$$\mathcal{L}^2\big(SL(n,\mathbb{Z}) \setminus \mathcal{H}^n\big) = \left\{ f \;\Big|\; \int\limits_{SL(n,\mathbb{Z})\setminus\mathcal{H}^n} f(z)\overline{f(z)}\, d^*z < \infty \right\}.$$

A Maass cusp form, or more generally a Maass form, is a smooth function on $\mathcal{H}^n$ which is invariant under the left action of $SL(n, \mathbb{Z})$, is an eigenfunction of all $GL(n, \mathbb{R})$-invariant differential operators and satisfies good growth conditions. A Maass cusp form satisfies an additional cuspidal condition.

The algebra of invariant differential operators can be realized as the center of the universal enveloping algebra of the Lie algebra of $GL(n, \mathbb{R})$, see [9]. For instance, we associate a differential operator to an element X in the Lie algebra of $GL(n, \mathbb{R})$, such that if f is a smooth function on $GL(n, \mathbb{R})$, then

$$(Xf)(g) = \frac{d}{dt} f\big(g \cdot \exp(tX)\big)\Big|_{t=0}.$$

In order to make the definition of a Maass form precise we need to define the I-function that parametrizes the eigenvalues of automorphic functions under the invariant differential operators.

For $z \in \mathcal{H}^n$ and $\lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{C}^{n-1}$, define the I-function as the product

$$I_\lambda(z) = \prod_{i=1}^{n-1} \prod_{j=1}^{n-1} y_i^{b_{i,j}\lambda_j},$$

where

$$b_{i,j} = \begin{cases} (n-i)j & \text{if } 1 \leqslant j \leqslant i, \\ (n-j)i & \text{if } i \leqslant j \leqslant n-1. \end{cases}$$

The function $I_\lambda(z)$ is an eigenfunction of all the $GL(n,\mathbb{R})$-invariant differential operators, see [9].

For any $GL(n,\mathbb{R})$-invariant differential operator D, let $\nu_D(\lambda)$ be the eigenvalue of the I-function acted on by D

$$DI_\lambda(z) = \nu_D(\lambda) I_\lambda(z). \tag{1}$$

Definition 2. For $\lambda \in \mathbb{C}^{n-1}$, a Maass cusp form of type λ for $SL(n,\mathbb{Z})$ is a smooth function $f(z)$ on $\mathcal{H}^n$ which satisfies the following conditions:

(1) $f(\gamma z) = f(z)$ for all $z \in \mathcal{H}^n$ and for any $\gamma \in SL(n,\mathbb{Z})$.
(2) $Df(z) = \nu_D(\lambda) f(z)$ for all $GL(n,\mathbb{R})$-invariant differential operators D where $\nu_D(\lambda)$ is given by (1).
(3) $f(z)$ has at most polynomial growth in each y_i as $y_i \to \infty$.
(4) $\int_{U_\pi(\mathbb{Z})\backslash U_\pi(\mathbb{R})} f(uz)\,du = 0$ for all π, where π is a partition of n and U_π is the group of unipotents associated to the partition π.

Note that by smooth, we mean a smooth function of the real variables $x_{i,j}$ and y_k.

Since an automorphic function is invariant under the subgroup $SL(n,\mathbb{Z})$ it will have a Fourier expansion. In order to study the L-functions associated to Maass cusp forms, we will need the thesis work of Jacquet, [13], which provides the basis functions and also the work of Shalika, [17], which provides the multiplicity-one theorem in this case.

Let U be the group of upper triangular unipotent matrices (1s on the diagonal). The character $\Phi(u)$, on the group U, is defined as a function of the $n-1$ elements of the supradiagonal of u, call these u_i, $i = (1,\dots,n-1)$, numbered from bottom to top. We define the character $\Phi(u)$ as follows:

$$\Phi(u) = \prod_{i=1}^{n-1} e(u_i) \quad \text{where } e(z) = e^{2\pi i z}.$$

Definition 3. For $z \in \mathcal{H}^n$ and $\lambda \in \mathbb{C}^{n-1}$, the Jacquet–Whittaker functions are defined by the absolutely convergent integral below when $\mathrm{Re}(\lambda_i) > \frac{1}{n}$ for all i and by meromorphic continuation to all $\lambda = (\lambda_1,\dots,\lambda_{n-1}) \in \mathbb{C}^{n-1}$

$$W_J(z,\lambda,\Phi) = \int_{U(\mathbb{R})} I_\lambda(wuz)\overline{\Phi(u)}\,d^*u.$$

Here the measure is $d^*u = \prod_{i,j=1,\, i<j}^{n} du_{i,j}$ and w is the long element of the Weyl group given by

$$w = \begin{pmatrix} & & & \pm 1 \\ & & 1 & \\ & \cdot^{\cdot^{\cdot}} & & \\ 1 & & & \end{pmatrix} \in SL(n,\mathbb{Z}). \tag{2}$$

We will usually suppress the dependence of the Jacquet–Whittaker function on λ and Φ and just write

$$W_J(z) = W_J(z, \lambda, \Phi).$$

For $\mathbf{r} = (r_1, \ldots, r_{n-1}) \in \mathbb{R}^{n-1}$ satisfying $r_i \neq 0$ for every $1 \leqslant i \leqslant n-1$, we can associate a matrix also denoted by r,

$$r = \begin{pmatrix} r_1 \ldots r_{n-1} & & & & \\ & r_1 \ldots r_{n-2} & & & \\ & & \ddots & & \\ & & & r_1 & \\ & & & & 1 \end{pmatrix} \in GL(n,\mathbb{R}). \tag{3}$$

It will hopefully be clear from the context whether we want the matrix or the vector.

The Fourier expansion of a Maass cusp form $f(z)$ of type λ in terms of Jacquet's Whittaker function is proved for example in [9]. We will just state the result here

$$f(z) = \sum_{\gamma \in \Gamma_\infty \backslash SL(n-1,\mathbb{Z})} \sum_{\substack{\mathbf{r}=(r_1,\ldots,r_{n-1}) \\ r_1 \in \mathbb{N} \\ r_2,\ldots,r_{n-1} \in \mathbb{Z}\backslash 0}} c_{\mathbf{r}} W_J\left(r \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z, \lambda, \Phi \right), \tag{4}$$

where $\Gamma_\infty = U(\mathbb{Z})$ and where we have used the notation of (3). The numbers $c_{\mathbf{r}} = c_{r_1,\ldots,r_{n-1}}$ are called the Fourier coefficients of the Maass cusp form $f(z)$.

1.1. L-functions and multiple Dirichlet series

Godement and Jacquet [8] defined the L-function associated to a Maass cusp form $f(z)$ on $\mathcal{H}^n$ as a Dirichlet series in one complex variable. For $\mathrm{Re}(s) \gg 1$, the following sum converges absolutely

$$L_{GJ}(f,s) = \sum_{r_1=1}^{\infty} \frac{c_{r_1,1,\ldots,1}}{r_1^s}.$$

The Fourier coefficients $c_{r_1,1,\ldots,1}$ come from the expansion in (4). The Godement–Jacquet L-functions have analytic continuation and a functional equation.

Using the full array of Fourier coefficients of a cusp form $f(z)$ found in (4) we can create the following multiple Dirichlet series which converges absolutely for $\mathbf{s} = (s_1, \ldots, s_{n-1}) \in \mathbb{C}^{n-1}$ and $\mathrm{Re}(s_i) \gg 1$ for all i

$$L_f(\mathbf{s}) = \sum_{r_1=1}^{\infty} \cdots \sum_{r_{n-1}=1}^{\infty} \frac{c_{r_1,\ldots,r_{n-1}}}{r_1^{s_1} \ldots r_{n-1}^{s_{n-1}}}. \tag{5}$$

We will call this sum, the full multiple Dirichlet series associated to the form. For $GL(3,\mathbb{R})$, Bump obtained the functional equation and meromorphic continuation of these series in [2].

Theorem 4 *(Bump). If $f(z)$ is a Maass cusp form on $\mathcal{H}^3$ and $\mathbf{s} = (s_1, s_2) \in \mathbb{C}^2$, then*

$$L_f(\mathbf{s}) = \frac{L_{GJ}(\tilde{f}, s_1) L_{GJ}(f, s_2)}{\zeta(s_1 + s_2)},$$

where $\tilde{f}$ is the contragredient cusp form defined as $\tilde{f}(z) = f({}^i z)$ where ^{i} is the involution defined on $\mathcal{H}^3$ by

$${}^i : z \to w\, {}^t z^{-1} w,$$

where w is the long element defined in (2). *Since the Godement–Jacquet L-functions are known to have analytic continuation and functional equation, it follows that $L_f(\mathbf{s})$ has meromorphic continuation and $\zeta(s_1 + s_2) L_f(\mathbf{s})$ has functional equations.*

In general, if one specializes the complex variables of the full multiple Dirichlet series to certain two-dimensional hyperplanes, the resulting multiple Dirichlet series will be related to standard L-functions. For instance, if $f(z)$ is a cusp form for $SL(4,\mathbb{Z})$, the following relation was proved by Bump and Friedberg in [3]:

$$L_f(s_1, s_2, s_1 + s_2) = \frac{L_{GJ}(f, s_1) L(s_2, f, \Lambda^2)}{\zeta(2s_2)},$$

where $L(s, f, \Lambda^2)$ is the exterior square L-function of the form.

2. Statement of main result

For the remainder of the paper, we will fix $\Gamma = SL(n,\mathbb{Z})$ and Γ_∞ to be the subgroup of Γ consisting of upper triangular unipotent matrices. Let $Z \subseteq \{\pm 1\}$ be the center of $SL(n,\mathbb{Z})$. Let Γ_1 be the corner group defined as the subgroup of Γ_∞ having 0s on the supradiagonal. For $n = 3$ the corner group is

$$\Gamma_1 = \left\{ \begin{pmatrix} 1 & 0 & a \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix} \,\middle|\, a \in \mathbb{Z} \right\}. \tag{6}$$

In order to define the kernel function, we need to change notation a little. We will need to complexify, as in the $GL(2)$ case. For any $z \in \mathcal{H}^n$, there are $n - 1$ elements of the supradiagonal of $x(z)$ and we will label them as x_k with ks numbered in increasing order from bottom to top. Then we define $z_k = x_k + iy_k$ for each $k \in \{1, \dots, n-1\}$. For example if $z \in \mathcal{H}^3$, as given below, then $z_1 = x_1 + iy_1$ and $z_2 = x_2 + iy_2$,

$$z = \begin{pmatrix} 1 & x_2 & x_{1,3} \\ & 1 & x_1 \\ & & 1 \end{pmatrix} \begin{pmatrix} y_1 y_2 & & \\ & y_1 & \\ & & 1 \end{pmatrix}. \tag{7}$$

The kernel function that produces the full multiple Dirichlet series of a Maass cusp form will arise from the following Poincaré series. The proofs follow.

Proposition 5. *Let* $\mathbf{v} = (v_1, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$ *and* $\mathbf{w} = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1}$ *satisfy* $\mathrm{Re}(v_k), \mathrm{Re}(w_k) \gg 1$ *for all k. Then, the Poincaré series defined by*

$$P(z, \mathbf{v}, \mathbf{w}) = \sum_{\gamma \in \Gamma_1 \backslash \Gamma / Z} \left[\prod_{k=1}^{n-1} \left(\frac{z_k}{i} \right)^{-w_k} y_k^{v_k + w_k} \right]_\gamma, \tag{8}$$

converges absolutely and uniformly on compact subsets of $\mathcal{H}^n$.

For example, the Poincaré series we will associate to $GL(3)$ is:

$$P(z, \mathbf{v}, \mathbf{w}) = \sum_{\gamma \in \Gamma_1 \backslash SL(3,\mathbb{Z})} \left(\frac{z_1}{i} \right)^{-w_1} \left(\frac{z_2}{i} \right)^{-w_2} y_1^{v_1 + w_1} y_2^{v_2 + w_2} \bigg|_\gamma,$$

with Γ_1 given in line (6) above.

Main Theorem. *Let* $f(z)$ *be a Maass cusp form on* $\mathcal{H}^n$. *Let* $\mathbf{v} = (v_1, \dots, v_{n-1}) \in \mathbb{C}^{n-1}$ *and* $\mathbf{w} = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1}$ *satisfy* $\mathrm{Re}(v_k), \mathrm{Re}(w_k) \gg 1$ *for all k. Let* $P(z, \mathbf{v}, \mathbf{w})$ *be the Poincaré series defined in* (8). *Then,*

$$\langle P(*, \mathbf{v}, \mathbf{w}), f \rangle = \left(\prod_{j=1}^{n-1} \frac{(2\pi)^{w_j}}{\Gamma(w_j)} \right) \overline{L_f(\bar{\mathbf{v}} - \mathbf{v_0})} G_J(\mathbf{v}, \mathbf{w}),$$

where $\mathbf{v_0}$ *is a constant depending in a simple way only on n, the full multiple Dirichlet series* $L_f(\mathbf{s})$ *is defined by* (5) *and* $G_J(\mathbf{v}, \mathbf{w})$ *is a function of* $\mathbf{v}$ *and* $\mathbf{w}$ *defined as*:

$$G_J(\mathbf{v}, \mathbf{w}) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} e^{-2\pi y_k} y_k^{v_k + w_k} \right] \overline{W}_J(y) \, d^*y. \tag{9}$$

On $GL(2)$ and $GL(3)$, we can use spectral theory due to Selberg and Langlands respectively to obtain the meromorphic continuation of the Poincaré series. For $n \geqslant 4$, Langlands' spectral theory [15] shows that the problem of obtaining the continuation of the full multiple Dirichlet series is equivalent to that of proving the continuation of the Poincaré series.

As mentioned in the Introduction, there are classical Poincaré series. Selberg in [16] studied Poincaré series on $GL(2)$ for each $m \in \mathbb{N}$. When the Poincaré series were integrated against a cusp form, he obtained the mth Fourier coefficient of the form. He proved that these Poincaré series were square-integrable.

The generalization to $GL(n)$ of Selberg's series was first done in Bump, Friedberg and Goldfeld [4] where they introduced classical Poincaré series for each $\mathbf{m} = (m_1, \dots, m_{n-1}) \in \mathbb{Z}^{n-1}$.

Definition 6 *(Bump, Friedberg, Goldfeld).* For $\mathbf{m} \in \mathbb{Z}^{n-1}$ let $U_{\mathbf{m}}(z, \mathbf{s})$ denote the classical Poincaré series on $\mathcal{H}^n$ defined by the following series which converges absolutely and uniformly on compact sets when $\mathbf{s} = (s_1, \dots, s_{n-1}) \in \mathbb{C}^{n-1}$ satisfies $\mathrm{Re}(s_k) \gg 1$ for all k:

$$U_{m_1,\dots,m_{n-1}}(z, \mathbf{s}) = \sum_{\gamma \in \Gamma_\infty \backslash \Gamma} \left[\prod_{k=1}^{n-1} e(m_k z_k) y_k^{s_k} \right]_\gamma .$$

When the $\mathbf{m}$th Poincaré series is used as the kernel of integration against a Maass cusp form, it essentially gives the $\mathbf{m}$th Fourier coefficient of the form. It is known that the classical Poincaré series are square-integrable and have meromorphic continuation to the entire $\mathbf{s}$ plane, see [6].

The following theorem shows how our Poincaré series are related to the classical ones.

Theorem 7. *The Poincaré series $P(z, \mathbf{v}, \mathbf{w})$ is the sum over the classical Poincaré series when* $\mathrm{Re}(v_k)$ *and* $\mathrm{Re}(w_k) \gg 1$ *for all k.*

2.1. Poincaré series

Let us begin with the proof of Proposition 5, the absolute convergence of the Poincaré series.

Lemma 8. *Let $\mathbf{w} = (w_1, \dots, w_{n-1}) \in \mathbb{C}^{n-1}$ satisfy* $\mathrm{Re}(w_k) \gg 1$ *for all k. Then*

$$f(z) = \sum_{m_1,\dots,m_{n-1} \in \mathbb{Z}} \left(\prod_{k=1}^{n-1} |z_k + m_k|^{-w_k} \right),$$

converges absolutely for $z \in \mathcal{H}^n$, and converges uniformly on each region S_v where

$$S_v = \left\{ z \in \mathcal{H}^n \text{ such that } \left| x_l(z) \right| \leqslant v \text{ for every } l \text{ from } 1 \text{ to } n-1 \right\}.$$

To prove Lemma 8, for each k, fix w_k such that $\mathrm{Re}(w_k) \gg 1$. We can use limit comparison between $f_k(z_k) := \sum_{m_k \in \mathbb{Z}} |z_k + m_k|^{-w_k}$ and the absolutely convergent series $\sum_{n>C} (n - C)^{-w_k}$ for some C depending on z_k. Thus, $f_k(z_k)$ converges pointwise when $\mathrm{Re}(w_k) \gg 1$. The product of absolutely convergent series converges absolutely so $f(z) = \prod_{k=1}^{n-1} f_k(z_k)$ converges absolutely when $\mathrm{Re}(w_k) \gg 1$ for all k.

Fix v to be a positive real number. For $z \in \mathcal{H}^n$ such that $|x_l(z)| \leqslant v$ for every l, we can use the Weierstrass test to compare $f_k(z_k)$ to $\sum_{n>v} (n - v)^{-w_k}$ for each k. Each sum $f_k(z_k)$ converges uniformly in the region given in the lemma.

Note that when $\mathrm{Re}(w) > 1$ and $z = x + iy$ where $|x| \leqslant \frac{1}{2}$ and $y > 0$, we have

$$\sum_{m\in\mathbb{Z}} \frac{1}{|z+m|^w} = \frac{1}{|z|^w} + \sum_{m\neq 0} \frac{1}{|z+m|^w} \leqslant \frac{1}{y^w} + \sum_{m\neq 0} \frac{1}{|z+m|^w}.$$

For each k, we can use the above remark to say that $|f_k(z_k)y_k^{w_k}|$ is bounded as $y_k \to 0^+$. Recall that $f(z) = \prod_{k=1}^{n-1} f_k(z_k)$. For any $a > 0$, the function $|f(z)\prod_{k=1}^{n-1} y_k^{w_k}|$ is bounded on the set of $z \in \mathcal{H}^n$ such that $y_l(z) < a$ for all $1 \leqslant l \leqslant n-1$.

For $\mathbf{w} = (w_1, \ldots, w_{n-1}) \in \mathbb{C}^{n-1}$ and $\mathbf{v} = (v_1, \ldots, v_{n-1}) \in \mathbb{C}^{n-1}$, define

$$I(z) = \prod_{k=1}^{n-1} y_k^{v_k+w_k},$$

where we suppress the dependence of $I(z)$ on $\mathbf{v}$ and $\mathbf{w}$.

Define

$$N(z) = \left(\prod_{k=1}^{n-1} y_k^{n-k}\right)^{-\frac{1}{n}}.$$

It can be shown by reduction theory, see [7] or [9], that for a fixed $z \in \mathcal{H}^n$, the function $N(z)$ achieves a positive minimum finitely many times on $SL(n,\mathbb{Z}) \cdot z$. Further, the minimum of $N(z)$ on $SL(n,\mathbb{Z}) \cdot z$ for fixed z is achieved at a point $w \in \mathcal{H}^n$ where $y_k(w) > \frac{\sqrt{3}}{2}$ for all $1 \leqslant k \leqslant n-1$.

It follows that for fixed $z \in \mathcal{H}^n$, there exists $A \gg \frac{\sqrt{3}}{2}$ such that $y_k(\gamma z) < A$ for all $\gamma \in SL(n,\mathbb{Z})$ and for all k from 1 to $n-1$.

Therefore, the function $|f(\gamma z)\prod_{k=1}^{n-1} y_k^{w_k}(\gamma z)|$ is bounded on the set of $\gamma \in SL(n,\mathbb{Z})$.

Using the above bound, we will prove that the series $\tilde{P}(z,\mathbf{v},\mathbf{w})$ defined by

$$\tilde{P}(z,\mathbf{v},\mathbf{w}) = \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \big[f(z)I(z)\big]_\gamma,$$

converges absolutely by comparing it to the minimal parabolic Eisenstein series. Then, by rearrangement of the terms of the absolutely convergent series $\tilde{P}$, we will show that our Poincaré series $P(z,\mathbf{v},\mathbf{w})$ converges absolutely

$$\begin{aligned}\big|\tilde{P}(z,\mathbf{v},\mathbf{w})\big| &\leqslant \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \big|f(\gamma z)I(\gamma z)\big| \\ &\leqslant c \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left[\prod_{k=1}^{n-1} y_k^{-w_k}(\gamma z) y_k^{v_k+w_k}(\gamma z)\right] \\ &= c \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left[\prod_{k=1}^{n-1} y_k^{v_k}\right]_\gamma.\end{aligned}$$

The last series is just the minimal parabolic Eisenstein series which is known to converge absolutely and uniformly on compact sets of $\mathcal{H}^n$, see [1], when $\mathrm{Re}(v_k) \gg 1$ for every k. Thus, $\tilde{P}$ converges absolutely. That $\tilde{P}$ converges uniformly on compact sets can be seen easily as follows.

For any $z_o \in \Gamma \setminus \mathcal{H}^n$ in a compact set C also contained in the fundamental domain $\Gamma \setminus \mathcal{H}^n$, we can modify the above arguments. By continuity, for sufficiently small C, there exists some $A \gg \frac{\sqrt{3}}{2}$ such that $y_k(\gamma z) < A$ for all k, for all $z \in C$ and for every $\gamma \in SL(n, \mathbb{Z})$. Therefore, the function $|f(\gamma z) \prod_{k=1}^{n-1} y_k^{w_k}(\gamma z)|$ is bounded on the set of $\gamma \in SL(n, \mathbb{Z})$ and $z \in C$.

To finish the argument suppose we have two sequences of functions such that $|g_i(z)| \leqslant |h_i(z)|$ for all i and $z \in C$ a compact set and suppose $\sum h_i(z)$ converges uniformly on C. Let $\epsilon > 0$. There exists $N > 0$ such that $|\sum_{i=k}^{m} h_i(z)| \leqslant \epsilon$ for all $z \in C$; $m, k > N$. Then, for all $z \in C$; $k, m > N$

$$\left|\sum_{i=k}^{m} g_i(z)\right| \leqslant \sum_{i=k}^{m} \left|g_i(z)\right| \leqslant \sum_{i=k}^{m} \left|h_i(z)\right| \leqslant \epsilon.$$

The last step to prove Proposition 5 is to show how P relates to $\tilde{P}$.

Let us recall the notation of some subgroups we will need here, namely Γ, Γ_1 and Γ_∞. Previously we had dropped the dependence of n in the notation but now we will need to use induction on n so we need to change notation slightly. We set $\Gamma = \Gamma(n)$ and $\Gamma_1 = \Gamma_1(n)$ and $\Gamma_\infty = \Gamma_\infty(n)$.

Thus $\Gamma(n) = SL(n, \mathbb{Z})$ and $\Gamma_1(n)$ is the corner group defined in Section 2.

If we fix a set of coset representatives of $\Gamma_1(n) \setminus \Gamma(n)$, $\Gamma_\infty(n) \setminus \Gamma(n)$ and of $\Gamma_1(n) \setminus \Gamma_\infty(n)$, then we have the following bijection of sets:

$$\Gamma_1(n) \setminus \Gamma(n) \quad \Leftrightarrow \quad \Gamma_1(n) \setminus \Gamma_\infty(n) \cdot \Gamma_\infty(n) \setminus \Gamma(n). \tag{10}$$

We can choose coset representatives of $\Gamma_1(n) \setminus \Gamma_\infty(n)$ with the following lemma. In the proof we will use an induction argument on n.

Lemma 9. *The map $\Phi_n : \Gamma_1(n) \setminus \Gamma_\infty(n) \leftrightarrow \mathbb{Z}^{n-1}$ defined*:

$$\Phi_n\big((x_{i,j})\big) \leftrightarrow \big((x_{i,i+1})\big),$$

is a well defined bijection of a matrix representative to its supradiagonal.

To see that Φ_n is a well-defined map, note that the action of $\Gamma_1(n)$ on $\Gamma_\infty(n)$ by matrix multiplication does not change the supradiagonal. It is clear that the map is surjective. Injectivity can be shown by induction. It is an easy calculation that Φ_3 is injective. Write $\gamma \in \Gamma_\infty(n)$ as below with $\alpha \in \Gamma_\infty(n-1)$

$$\gamma = \begin{pmatrix} \alpha & u \\ 0 & 1 \end{pmatrix}.$$

Now if $\gamma, \gamma' \in \Gamma_\infty(n)$ have the same supradiagonal then $\alpha, \alpha' \in \Gamma_\infty(n-1)$ have the same supradiagonal. By induction, there exists a matrix $\beta \in \Gamma_1(n-1)$ such that $\beta \cdot \alpha = \alpha'$. Now consider the matrix

$$\delta = \begin{pmatrix} \beta & u' - \beta u \\ 0 & 1 \end{pmatrix}.$$

We can see that $\delta\gamma = \gamma'$ and $\delta \in \Gamma_1(n)$ because $\gamma, \gamma' \in \Gamma_\infty(n)$ have the same supradiagonal. So the lemma is proved.

Using the lemma, we can choose a set of coset representatives of $\Gamma_1 \setminus \Gamma_\infty$ to be all the matrices with 1s on the diagonal and integral entries on the supradiagonal and 0s everywhere else.

By rearrangement of the series $\tilde{P}(z, \mathbf{v}, \mathbf{w})$ for $\mathrm{Re}(v_k), \mathrm{Re}(w_k) \gg 1$, we see that

$$\begin{aligned}
\tilde{P}(z, \mathbf{v}, \mathbf{w}) &= \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \sum_{m_1, \ldots, m_{n-1} \in \mathbb{Z}} \left[\prod_{k=1}^{n-1} y_k(\gamma z)^{v_k + w_k} \left| z_k(\gamma z) + m_k \right|^{-w_k} \right] \\
&= \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \sum_{\alpha \in \Gamma_1 \setminus \Gamma_\infty} \left[\prod_{k=1}^{n-1} y_k(\gamma z)^{v_k + w_k} \left| z_k(\alpha\gamma z) \right|^{-w_k} \right] \\
&= \sum_{\gamma \in \Gamma_\infty \setminus \Gamma} \sum_{\alpha \in \Gamma_1 \setminus \Gamma_\infty} \left[\prod_{k=1}^{n-1} y_k(\alpha\gamma z)^{v_k + w_k} \left| z_k(\alpha\gamma z) \right|^{-w_k} \right] \\
&= \sum_{\beta \in \Gamma_1 \setminus \Gamma} \left[\prod_{k=1}^{n-1} y_k(z)^{v_k + w_k} \left| z_k(z) \right|^{-w_k} \right]_\beta,
\end{aligned}$$

where we have used (10) and the invariance of $I(z)$ under $\alpha \in \Gamma_\infty$.

Recall that

$$P(z, \mathbf{v}, \mathbf{w}) = \sum_{\gamma \in \Gamma_1 \setminus \Gamma / Z} \left[\prod_{k=1}^{n-1} \left(\frac{z_k}{i} \right)^{-w_k} y_k^{v_k + w_k} \right]_\gamma. \tag{11}$$

Since

$$\begin{aligned}
\left| \sum_{\gamma \in \Gamma_1 \setminus \Gamma} \prod_{k=1}^{n-1} \left(\frac{z_k}{i} \right)^{-w_k} y_k^{v_k + w_k} \right|_\gamma &\leqslant \sum_{\gamma \in \Gamma_1 \setminus \Gamma} \prod_{k=1}^{n-1} \left| \left(\frac{z_k}{i} \right)^{-w_k} y_k^{v_k + w_k} \right|_\gamma, \\
&= \tilde{P}(z, \mathbf{v}, \mathbf{w}),
\end{aligned}$$

we have the absolute and uniform convergence of $P(z, \mathbf{v}, \mathbf{w})$ on compact sets when $\mathrm{Re}(w_k), \mathrm{Re}(v_k) \gg 1$. Proposition 5 is proved.

Now, we will prove Theorem 7, namely that our Poincaré series are related to the classical Poincaré series of Selberg, Bump, Friedberg and Goldfeld.

Since the Poincaré series converges absolutely for $\mathrm{Re}(v_k), \mathrm{Re}(w_k) \gg 1$ for all k, we can rearrange the series in this region. Let $|Z|$ be the order of the center.

$$P(z,\mathbf{v},\mathbf{w}) = |Z| \sum_{\delta\in\Gamma_1\backslash\Gamma} \left[\prod_{k=1}^{n-1}\left(\frac{z_k(z)}{i}\right)^{-w_k} y_k(z)^{v_k+w_k}\right]_\delta$$
$$= |Z| \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \sum_{\alpha\in\Gamma_1\backslash\Gamma_\infty} \left[\prod_{k=1}^{n-1}\left(\frac{z_k(\alpha\gamma z)}{i}\right)^{-w_k} y_k(\alpha\gamma z)^{v_k+w_k}\right].$$

Using Lemma 9, we can again choose a set of coset representatives of $\Gamma_1 \setminus \Gamma_\infty$ to be all the matrices with 1s on the diagonal and integral entries on the supradiagonal and 0s everywhere else

$$P(z,\mathbf{v},\mathbf{w}) = |Z| \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \sum_{r_1,\dots,r_{n-1}\in\mathbb{Z}} \left[\prod_{k=1}^{n-1}\left(\frac{z_k(\gamma z)+r_k}{i}\right)^{-w_k} y_k(\gamma z)^{v_k+w_k}\right]$$
$$= |Z| \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left[\sum_{r_1,\dots,r_{n-1}\in\mathbb{Z}} \prod_{k=1}^{n-1}\left(\frac{z_k(\gamma z)+r_k}{i}\right)^{-w_k}\right]\left[\prod_{i=1}^{n-1} y_i(\gamma z)^{v_i+w_i}\right]$$
$$= |Z| \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left[\prod_{k=1}^{n-1}\sum_{r_k\in\mathbb{Z}}\left(\frac{z_k(\gamma z)+r_k}{i}\right)^{-w_k}\right]\left[\prod_{i=1}^{n-1} y_i(\gamma z)^{v_i+w_i}\right]$$
$$= |Z| \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left[\prod_{k=1}^{n-1}\sum_{r_k\in\mathbb{Z}}\left(\frac{z_k(z)+r_k}{i}\right)^{-w_k}\right]\left[\prod_{i=1}^{n-1} y_i(z)^{v_i+w_i}\right]\Bigg|_\gamma.$$

Note that the above inner sums converge absolutely when $\mathrm{Re}(w_k) > 1$ for each k. In fact, we have the Lipschitz Summation Formula [14], valid for $\mathrm{Re}(w) > 1$ when $z = x + iy$ and $y > 0$

$$\sum_{r\in\mathbb{Z}}\left(\frac{z+r}{i}\right)^{-w} = \frac{(2\pi)^w}{\Gamma(w)}\sum_{m=1}^{\infty} m^{w-1} e(mz). \tag{12}$$

Let us use the summation formula in our calculation

$$P(z,\mathbf{v},\mathbf{w}) = |Z|\left(\prod_{j=1}^{n-1}\frac{(2\pi)^{w_j}}{\Gamma(w_j)}\right)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left[\prod_{k=1}^{n-1}\sum_{m_k=1}^{\infty} m_k^{w_k-1} e(m_k z_k)\right]\left[\prod_{i=1}^{n-1} y_i^{v_i+w_i}\right]\Bigg|_\gamma$$
$$= |Z|\left(\prod_{j=1}^{n-1}\frac{(2\pi)^{w_j}}{\Gamma(w_j)}\right)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\left[\prod_{k=1}^{n-1}\sum_{m_k=1}^{\infty} m_k^{w_k-1} e(m_k z_k) y_k^{v_k+w_k}\right]_\gamma.$$

As in the proof of the absolute convergence of $\tilde{P}$, we can show the above multiple sums converge absolutely when $\mathrm{Re}(w_k)$ and $\mathrm{Re}(v_k) \gg 1$ by comparing the sum of the absolute values to the minimal parabolic Eisenstein series

$$P(z,\mathbf{v},\mathbf{w}) = |Z|\left(\prod_{j=1}^{n-1}\frac{(2\pi)^{w_j}}{\Gamma(w_j)}\right)\sum_{\gamma\in\Gamma_\infty\backslash\Gamma}\sum_{m_1,\dots,m_{n-1}\in\mathbb{N}}\left[\prod_{k=1}^{n-1} m_k^{w_k-1} e(m_k z_k) y_k^{v_k+w_k}\right]_\gamma.$$

We will now interchange the order of summation when $\mathrm{Re}(v_k), \mathrm{Re}(w_k) \gg 1$

$$P(z,\mathbf{v},\mathbf{w}) = |Z| \left(\prod_{j=1}^{n-1} \frac{(2\pi)^{w_j}}{\Gamma(w_j)} \right) \sum_{m_1,\ldots,m_{n-1}\in\mathbb{N}} \sum_{\gamma\in\Gamma_\infty\backslash\Gamma} \left[\prod_{k=1}^{n-1} m_k^{w_k-1} e(m_k z_k) y_k^{v_k+w_k} \right]_\gamma$$

$$= |Z| \left(\prod_{j=1}^{n-1} \frac{(2\pi)^{w_j}}{\Gamma(w_j)} \right) \sum_{m_1,\ldots,m_{n-1}\in\mathbb{N}} \left(\prod_{k=1}^{n-1} m_k^{w_k-1} \right) U_{m_1,\ldots,m_{n-1}}(z,\mathbf{v}+\mathbf{w}).$$

Note that $U_{m_1,\ldots,m_{n-1}}(z,\mathbf{s})$ denotes the classical Poincaré series from Definition 6.

2.2. The kernel for L-functions

Main Theorem. *Let $f(z)$ be a Maass cusp form on $\mathcal{H}^n$. Let $\mathbf{v} = (v_1,\ldots,v_{n-1}) \in \mathbb{C}^{n-1}$ and $\mathbf{w} = (w_1,\ldots,w_{n-1}) \in \mathbb{C}^{n-1}$ satisfy $\mathrm{Re}(v_k), \mathrm{Re}(w_k) \gg 1$ for all k. Let $P(z,\mathbf{v},\mathbf{w})$ be the Poincaré series defined in* (8)*. Then,*

$$\langle P(*,\mathbf{v},\mathbf{w}), f \rangle = \left(\prod_{j=1}^{n-1} \frac{(2\pi)^{w_j}}{\Gamma(w_j)} \right) \overline{L_f(\bar{\mathbf{v}} - \mathbf{v}_0)} G_J(\mathbf{v},\mathbf{w}),$$

where the constant $\mathbf{v}_0$ depends in a simple way only on n, the full multiple Dirichlet series $L_f(\mathbf{s})$ is defined by (5) *and $G_J(\mathbf{v},\mathbf{w})$ is a function of $\mathbf{v}$ and $\mathbf{w}$ defined as*:

$$G_J(\mathbf{v},\mathbf{w}) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} e^{-2\pi y_k} y_k^{v_k+w_k} \right] \overline{W}_J(y)\, d^*y. \tag{13}$$

Recall that $\Gamma = SL(n,\mathbb{Z})$ and Γ_1 is the corner group. Fix $\mathbf{v},\mathbf{w} \in \mathbb{C}^{n-1}$ in the region of absolute convergence of the Poincaré series. Since $f(z)$ is a cusp form and the Poincaré series $P(z,\mathbf{v},\mathbf{w})$ has at most slow growth in y_i as $y_i \to \infty$, the inner product $\langle P(*,\mathbf{v},\mathbf{w}), f\rangle$ converges absolutely

$$\langle P, f\rangle = \int_{\Gamma\backslash\mathcal{H}^n} P(z,\mathbf{v},\mathbf{w}) \overline{f(z)}\, d^*z$$

$$= \int_{\Gamma\backslash\mathcal{H}^n} \sum_{\gamma\in\Gamma_1\backslash\Gamma} \left[\prod_{k=1}^{n-1} \left(\frac{z_k(\gamma z)}{i} \right)^{-w_k} y_k(\gamma z)^{v_k+w_k} \right] \overline{f(z)}\, d^*z$$

$$= \sum_{\gamma\in\Gamma_1\backslash\Gamma} \int_{\Gamma\backslash\mathcal{H}^n} \left[\prod_{k=1}^{n-1} \left(\frac{z_k(\gamma z)}{i} \right)^{-w_k} y_k(\gamma z)^{v_k+w_k} \right] \overline{f(z)}\, d^*z$$

$$= \int_{\Gamma_1\backslash\mathcal{H}^n} \left[\prod_{k=1}^{n-1} \left(\frac{z_k(z)}{i} \right)^{-w_k} y_k(z)^{v_k+w_k} \right] \overline{f(z)}\, d^*z.$$

We are integrating over a fundamental domain for the action of Γ_1 on $\mathcal{H}^n$ and such a fundamental domain can be obtained as the disjoint union

$$\Gamma_1 \setminus \mathcal{H}^n = \bigcup_{\gamma \in \Gamma_1 \setminus \Gamma_\infty} \gamma \cdot \left(\Gamma_\infty \setminus \mathcal{H}^n\right).$$

By absolute convergence and the dominated convergence theorem, we can continue the calculation as follows:

$$\begin{aligned}
\langle P, f\rangle &= \sum_{\gamma \in \Gamma_1 \setminus \Gamma_\infty} \int_{\gamma\cdot(\Gamma_\infty \setminus \mathcal{H}^n)} \left[\prod_{k=1}^{n-1} \left(\frac{z_k(z)}{i}\right)^{-w_k} y_k^{v_k+w_k}(z)\right] \overline{f(z)}\, d^*z \\
&= \sum_{\gamma \in \Gamma_1 \setminus \Gamma_\infty} \int_{\Gamma_\infty \setminus \mathcal{H}^n} \left[\prod_{k=1}^{n-1} \left(\frac{z_k(\gamma z)}{i}\right)^{-w_k} y_k^{v_k+w_k}(\gamma z)\right] \overline{f(\gamma z)}\, d^*\gamma z \\
&= \sum_{\gamma \in \Gamma_1 \setminus \Gamma_\infty} \int_{\Gamma_\infty \setminus \mathcal{H}^n} \left[\prod_{k=1}^{n-1} \left(\frac{z_k(\gamma z)}{i}\right)^{-w_k} y_k^{v_k+w_k}(z)\right] \overline{f(z)}\, d^*z.
\end{aligned}$$

We can choose coset representatives of $\Gamma_1 \setminus \Gamma_\infty$ with Lemma 9 to be matrices with 1s on the diagonal and integral entries on the supradiagonal and 0s everywhere else. Picking this set of coset representatives, let us continue the calculation of $\langle P, f\rangle$

$$\begin{aligned}
\langle P, f\rangle &= \sum_{\gamma \in \Gamma_1 \setminus \Gamma_\infty} \int_{\Gamma_\infty \setminus \mathcal{H}^n} \left[\prod_{k=1}^{n-1} \left(\frac{z_k(\gamma z)}{i}\right)^{-w_k} y_k^{v_k+w_k}(z)\right] \overline{f(z)}\, d^*z \\
&= \sum_{r_1,\dots,r_{n-1} \in \mathbb{Z}} \int_{\Gamma_\infty \setminus \mathcal{H}^n} \left[\prod_{k=1}^{n-1} \left(\frac{z_k + r_k}{i}\right)^{-w_k} y_k^{v_k+w_k}\right] \overline{f(z)}\, d^*z \\
&= \int_{\Gamma_\infty \setminus \mathcal{H}^n} \sum_{r_1,\dots,r_{n-1} \in \mathbb{Z}} \left[\prod_{k=1}^{n-1} \left(\frac{z_k + r_k}{i}\right)^{-w_k} y_k^{v_k+w_k}\right] \overline{f(z)}\, d^*z.
\end{aligned}$$

We have used absolute convergence to interchange the summation and integration. When $\mathrm{Re}(w_k) > 1$ for all k, the sum over the finite product is the finite product of the convergent sums. We can factor the y_ks and the $f(z)$ out of the convergent sums,

$$\langle P, f\rangle = \int_{\Gamma_\infty \setminus \mathcal{H}^n} \left[\prod_{k=1}^{n-1} \sum_{r_k \in \mathbb{Z}} \left(\frac{z_k + r_k}{i}\right)^{-w_k}\right] \left[\prod_{j=1}^{n-1} y_j^{v_j+w_j}\right] \overline{f(z)}\, d^*z.$$

Lemma 10. *A fundamental domain for the action of Γ_∞ on $\mathcal{H}^n$ is given by*

$$\Gamma_\infty \setminus \mathcal{H}^n = \left\{ z \in \mathcal{H}^n \;\middle|\; 0 \leqslant x_{i,j} < 1 \text{ and } y_l > 0 \text{ for every } 1 \leqslant i < j \leqslant n,\ 1 \leqslant l \leqslant n-1 \right\}.$$

For example when $n = 3$:

$$\int\limits_{\Gamma_\infty\backslash\mathcal{H}^n} = \int_0^\infty\int_0^\infty\int_0^1\int_0^1\int_0^1.$$

This simple lemma can be proved by induction on n. We will skip the proof.

To simplify the calculation, let $g(\mathbf{w})$ be the following product of gamma functions

$$g(\mathbf{w}) = \prod_{j=1}^{n-1} \frac{(2\pi)^{w_j}}{\Gamma(w_j)}. \tag{14}$$

We will now use the Lipschitz Summation Formula (12) $n-1$ times in our calculation of the inner product. Since we have $\mathrm{Re}(w_k) > 1$ for each k, the inner sums converge uniformly,

$$\begin{aligned}
\langle P, f\rangle &= g(\mathbf{w}) \cdot \int\limits_{\Gamma_\infty\backslash\mathcal{H}^n} \left[\prod_{k=1}^{n-1} \sum_{m_k=1}^{\infty} m_k^{w_k-1} e(m_k z_k)\right]\left[\prod_{j=1}^{n-1} y_j^{v_j+w_j}\right]\overline{f(z)}\, d^*z, \\
&= g(\mathbf{w}) \cdot \int\limits_{\Gamma_\infty\backslash\mathcal{H}^n} \left[\prod_{k=1}^{n-1} \sum_{m_k=1}^{\infty} m_k^{w_k-1} y_k^{v_k+w_k} e(m_k z_k)\right]\overline{f(z)}\, d^*z.
\end{aligned}$$

Since the inner sums converge uniformly and since $f(z)$ is smooth, we are integrating a smooth function of the real variables $x_{i,j}$ and y_l, over the fundamental domain given by Lemma 10. To show the absolute convergence of the integral we need to check the behavior of the integrand as each $y_l \to 0$ and $y_l \to \infty$. We know $f(z)$ is a cusp form, so it is bounded and in fact decays rapidly as each $y_l \to \infty$. The inner sums also decay rapidly as $y_l \to \infty$, for any l since $m_l \geqslant 1$. As each $y_l \to 0$, the inner sums are characterized by $y_l^{v_l}$ when $\mathrm{Re}(v_l), \mathrm{Re}(w_l) > 1$. We will rewrite the last line and then interchange the integration and summation

$$\begin{aligned}
\langle P, f\rangle &= g(\mathbf{w}) \cdot \int\limits_{\Gamma_\infty\backslash\mathcal{H}^n} \sum_{m_1,\dots,m_{n-1}\in\mathbb{N}} \left[\prod_{k=1}^{n-1} m_k^{w_k-1} y_k^{v_k+w_k} e(m_k z_k)\right]\overline{f(z)}\, d^*z \\
&= g(\mathbf{w}) \sum_{m_1,\dots,m_{n-1}\in\mathbb{N}} \cdot \int\limits_{\Gamma_\infty\backslash\mathcal{H}^n} \left[\prod_{k=1}^{n-1} m_k^{w_k-1} y_k^{v_k+w_k} e(m_k z_k)\right]\overline{f(z)}\, d^*z \\
&= g(\mathbf{w}) \sum_{m_1,\dots,m_{n-1}\in\mathbb{N}} m_1^{w_1-1}\dots m_{n-1}^{w_{n-1}-1} \int\limits_{\Gamma_\infty\backslash\mathcal{H}^n} \left[\prod_{k=1}^{n-1} y_k^{v_k+w_k} e(m_k z_k)\right]\overline{f(z)}\, d^*z \\
&= g(\mathbf{w}) \sum_{m_1,\dots,m_{n-1}\in\mathbb{N}} m_1^{w_1-1}\dots m_{n-1}^{w_{n-1}-1} \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} e^{-2\pi m_k y_k} y_k^{v_k+w_k}\right] I_f(\mathbf{m}, y)\, d^*y.
\end{aligned}$$

In the last line we have $I_f(\mathbf{m}, y)$, a Whittaker function, defined as follows.

Definition 11. For $\mathbf{m} = (m_1, \dots, m_{n-1}) \in \mathbb{N}^{n-1}$, let

$$I_f(\mathbf{m}, y) = \int_0^1 \cdots \int_0^1 \left[\prod_{k=1}^{n-1} e(m_k x_k)\right] \overline{f(z)}\, d^* x.$$

The multiplicity one theorem of Shalika tells us that $I_f(\mathbf{m}, y)$ is a scalar multiple of the Jacquet–Whittaker function. We will show that the constant is exactly that which appears in the Fourier expansion of $f(z)$. Let us assume the following proposition, which will be proved below, and finish the proof of the theorem. Note that the Jacquet–Whittaker function has rapid decay as $y_l \to \infty$ for any l.

Proposition 12. *For* $\mathbf{m} \in \mathbb{N}^{n-1}$,

$$I_f(\mathbf{m}, y) = \bar{c}_{\mathbf{m}} \overline{W}_J(my),$$

where $^-$ *denotes complex conjugation and* y *is the matrix obtained from* $\mathbf{y} = (y_1, \dots, y_{n-1})$. *Thus the matrix* my *is the product of the matrix obtained from* $\mathbf{m} = (m_1, \dots, m_{n-1})$ *and the matrix obtained from* $\mathbf{y} = (y_1, \dots, y_{n-1})$.

The inner product becomes

$$\begin{aligned}
\langle P, f\rangle &= g(\mathbf{w}) \sum_{m_1,\dots,m_{n-1}\in\mathbb{N}} m_1^{w_1-1} \dots m_{n-1}^{w_{n-1}-1} \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} e^{-2\pi m_k y_k} y_k^{v_k+w_k}\right] I_f(\mathbf{m}, y)\, d^* y \\
&= g(\mathbf{w}) \sum_{m_1,\dots,m_{n-1}\in\mathbb{N}} \bar{c}_{\mathbf{m}} m_1^{w_1-1} \dots m_{n-1}^{w_{n-1}-1} \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} e^{-2\pi m_k y_k} y_k^{v_k+w_k}\right] \overline{W}_J(my)\, d^* y.
\end{aligned}$$

The matrix $\tau \equiv my \bmod (O(n,\mathbb{R})Z(n,\mathbb{R}))$ has as its Iwasawa coordinates $y_k(\tau) = m_k y_k$. Make the change of variables $m_k y_k \to y_k$ for every k. Recall,

$$d^* y = \prod_{k=1}^{n-1} y_k^{-k(n-k)-1}\, dy_k.$$

Continuing the calculation:

$$\langle P, f\rangle = g(\mathbf{w}) \sum_{m_1,\dots,m_{n-1}\in\mathbb{N}} \frac{\bar{c}_{\mathbf{m}}}{[\prod_{k=1}^{n-1} m_k^{v_k+1-k(n-k)}]} \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} e^{-2\pi y_k} y_k^{v_k+w_k}\right] \overline{W}_J(y)\, d^* y.$$

We notice that the integral no longer has a dependence on $\mathbf{m}$. Let us make the following two definitions:

$$G_J(\mathbf{v}, \mathbf{w}) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} e^{-2\pi y_k} y_k^{v_k+w_k}\right] \overline{W}_J(y)\, d^* y, \tag{15}$$

$$\mathbf{v}_0 = (v_{0,1}, \ldots, v_{0,n-1}) \quad \text{with } v_{0,k} = k(n-k) - 1. \tag{16}$$

The integral defining the function $G_J(\mathbf{v}, \mathbf{w})$ converges absolutely when $\mathrm{Re}(w_k), \mathrm{Re}(v_k) > 1$ for all k because the Jacquet–Whittaker function has rapid decay as $y_k \to \infty$ for any k. Also it is known that $L_f(\mathbf{v})$ converges absolutely when $\mathrm{Re}(v_k) \gg 1$ for all k,

$$\begin{aligned}\langle P, f\rangle &= g(\mathbf{w}) \sum_{m_1,\ldots,m_{n-1}\in\mathbb{N}} \frac{\bar{c}_{\mathbf{m}}}{[\prod_{k=1}^{n-1} m_k^{v_k+1-k(n-k)}]} \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} e^{-2\pi y_k} y_k^{v_k+w_k}\right] \overline{W}_J(y)\, d^*y \\ &= g(\mathbf{w}) \cdot \overline{L_f(\bar{\mathbf{v}} - \mathbf{v}_0)} G_J(\mathbf{v}, \mathbf{w}).\end{aligned}$$

Now it remains to analyze $G_J(\mathbf{v}, \mathbf{w})$. Here, we will need to use the dependence of $W_J(z) = W_J(z, \lambda, \Phi)$ on λ and Φ. Let $\tilde{W}_J(\mathbf{s}, \lambda, \Phi)$ denote the multiple Mellin transform of $W_J(y, \lambda, \Phi)$ defined below for $\mathrm{Re}(s_k)$ sufficiently large

$$\tilde{W}_J(\mathbf{s}, \lambda, \Phi) = \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} y_k^{s_k} W_J(y, \lambda, \Phi)\right] \prod_{i=1}^{n-1} \frac{dy_i}{y_i}. \tag{17}$$

In the definition of $G_J(\mathbf{v}, \mathbf{w})$, we will substitute the expansion of the exponential function for each k, and then use $\overline{W}_J(y, \lambda, \Phi) = W_J(y, \bar{\lambda}, \overline{\Phi})$,

$$\begin{aligned}G_J(\mathbf{v}, \mathbf{w}) &= \sum_{c_1,\ldots,c_{n-1}=0}^{\infty} \left(\prod_{l=1}^{n-1} \frac{(-2\pi)^{c_l}}{c_l!}\right) \int_0^\infty \cdots \int_0^\infty \left[\prod_{k=1}^{n-1} y_k^{v_k+w_k+c_k} \overline{W}_J(y)\right] d^*y \\ &= \sum_{\substack{c_1,\ldots,c_{n-1}=0 \\ c=(c_1,\ldots,c_{n-1})}}^{\infty} \left(\prod_{l=1}^{n-1} \frac{(-2\pi)^{c_l}}{c_l!}\right) \tilde{W}_J\big(\mathbf{s}(\mathbf{c}, \mathbf{v}, \mathbf{w}), \bar{\lambda}, \overline{\Phi}\big),\end{aligned}$$

where $\mathbf{s}(\mathbf{c}, \mathbf{v}, \mathbf{w}) = (s_1, \ldots, s_{n-1})$ and $s_k = v_k + w_k + c_k - k(n-k)$ for each k.

It is proved in [6] that the function $\tilde{W}_J(\mathbf{s}, \lambda, \Phi)$ has at most polynomial growth in the variable $\mathbf{s}$ and has meromorphic continuation to all $\mathbf{s} \in \mathbb{C}^{n-1}$. Thus $G_J(\mathbf{v}, \mathbf{w})$ is the sum of meromorphic functions.

Now for a proof of Proposition 12. In the definition of $I_f(\mathbf{m}, y)$ in line (11) we will substitute the Fourier expansion of $f(z)$,

$$I_f(\mathbf{m}, y) = \sum_{\gamma\in\Gamma_\infty\backslash SL(n-1,\mathbb{Z})} \sum_{\substack{\mathbf{r}=(r_1,\ldots,r_{n-1}) \\ r_1\in\mathbb{N} \\ r_2,\ldots,r_{n-1}\in\mathbb{Z}\backslash 0}} \overline{c_{\mathbf{r}}} \int_0^1 \cdots \int_0^1 \left(\prod_{k=1}^{n-1} e(m_k x_k)\right) \overline{W}_J\left(r \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z\right) d^*x.$$

In the definition of the Whittaker function, write $z = xy$ and make the change of variables $ux \to u$ to prove that the dependence on z separates into dependence on x and y. Here we are using the fact that the measure d^*u is left and right U-invariant. Thus,

$$W_J(z) = \Phi(x) W_J(y). \tag{18}$$

Let $\tau = x(\tau)y(\tau)$ be the Iwasawa coordinates of $\left(r\left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right)z\right)$. Then,

$$W_J(\tau) = \left(\prod_{k=1}^{n-1} e\big(x_k(\tau)\big)\right) W_J\big(y(\tau)\big).$$

It follows that

$$I_f(\mathbf{m}, y) = \sum_{\gamma \in \Gamma_\infty \backslash SL(n-1,\mathbb{Z})} \sum_{\substack{\mathbf{r}=(r_1,\dots,r_{n-1})\\ r_1 \in \mathbb{N}\\ r_2,\dots,r_{n-1} \in \mathbb{Z}\backslash 0}} \overline{c_{\mathbf{r}}} \int_0^1 \cdots \int_0^1 \left[\prod_{k=1}^{n-1} e(m_k x_k) e\big(-x_k(\tau)\big)\right] \overline{W}_J\big(y(\tau)\big)\, d^*x. \tag{19}$$

To simplify this further, we will need to know somewhat explicitly the Iwasawa coordinates of τ. First, we need to introduce some notation. For any $z \in \mathcal{H}^m$, let u_m be the mth column vector of dimension $m-1$ of the $x(z)$ matrix. Specifically we can write $z \in \mathcal{H}^m$ as below where $z' \in \mathcal{H}^{m-1}$

$$z = \begin{pmatrix} z' & u_m \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1 \mathbb{I}_{m-1} & \\ & 1 \end{pmatrix}. \tag{20}$$

We want to compute $u_n(\tau)$. Our strategy is to first find the Iwasawa decomposition of $\left(\begin{smallmatrix}\gamma & 0\\ 0 & 1\end{smallmatrix}\right)z$ where $z \in \mathcal{H}^n$ is a matrix in the form $z = xy$ and $\gamma \in SL(n-1,\mathbb{Z})$. Then we will find the Iwasawa decomposition of rz where r is the diagonal matrix given by (3) for $\mathbf{r} = (r_1, \dots, r_{n-1})$ and z is again a matrix in the form $z = xy$.

Lemma 13. *Given $\gamma \in GL(m-1,\mathbb{Z})$ and $z = xy \in \mathcal{H}^m$, let $\eta \in \mathcal{H}^m$ be defined by*

$$\eta \equiv \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z \mod \big(O(m,\mathbb{R})Z(m,\mathbb{R})\big).$$

Then $u_m(\eta) = \gamma u_m(z)$.

The proof follows:

$$\begin{aligned}
\eta &\equiv \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z \mod \big(O(m,\mathbb{R})Z(m,\mathbb{R})\big)\\
&\equiv \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & u_m(z) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1(z) \mathbb{I}_{m-1} & \\ & 1 \end{pmatrix}\\
&\equiv \begin{pmatrix} \gamma z' & \gamma u_m(z) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1(z) \mathbb{I}_{m-1} & \\ & 1 \end{pmatrix}.
\end{aligned}$$

We can find $k \in O(m-1,\mathbb{R})$, $d \in Z(m-1,\mathbb{R})$ and $z'' \in \mathcal{H}^{m-1}$ so that $\gamma z' = z'' kd$,

$$\begin{aligned}\eta &\equiv \begin{pmatrix} z''kd & \gamma u_m(z) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} y_1(z)\mathbb{I}_{m-1} & \\ & 1 \end{pmatrix} \mod \big(O(m,\mathbb{R})Z(m,\mathbb{R})\big) \\ &\equiv \begin{pmatrix} z'' & \gamma u(z) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} dy_1(z)\mathbb{I}_{m-1} & \\ & 1 \end{pmatrix} \cdot \begin{pmatrix} k & 0 \\ 0 & 1 \end{pmatrix}.\end{aligned}$$

It follows,

$$\eta \equiv \begin{pmatrix} z'' & \gamma u(z) \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} |d|y_1(z)\mathbb{I}_{m-1} & \\ & 1 \end{pmatrix} \mod \big(O(m,\mathbb{R})Z(m,\mathbb{R})\big). \tag{21}$$

The right-hand side is in Iwasawa form. By uniqueness, the lemma is proved. Note also from this proof that the only η coordinates that depend on $u_m(z)$ are the coordinates of $u_m(\eta)$.

Lemma 14. *Given* $\mathbf{r} = (r_1, \ldots, r_{m-1}) \in (\mathbb{Z} \setminus 0)^{m-1}$, *and* $z = xy \in \mathcal{H}^m$, *let* $\eta \in \mathcal{H}^m$ *be defined by*

$$\eta \equiv rz \mod \big(O(m,\mathbb{R})Z(m,\mathbb{R})\big),$$

where r *is the diagonal matrix associated to* $\mathbf{r}$ *by* (3). *Then*

$$u_m(\eta) = \begin{pmatrix} r_1 \ldots r_{m-1} & & & \\ & \ddots & & \\ & & r_1r_2 & \\ & & & r_1 \end{pmatrix} u_m(z).$$

Note that matrix multiplication gives the following identity

$$\begin{aligned}&\begin{pmatrix} R_1 & & \\ & \ddots & \\ & & R_m \end{pmatrix} \cdot \big((u_{i,j})\big) \cdot \begin{pmatrix} Y_1 & & \\ & \ddots & \\ & & Y_m \end{pmatrix} \\ &\quad = \left(\left(\frac{R_i u_{i,j}}{R_j}\right)_{i,j}\right) \begin{pmatrix} R_1Y_1 & & \\ & \ddots & \\ & & R_mY_m \end{pmatrix}.\end{aligned}$$

From this we see that the following is true, where we let $r_{-1} = 1$:

$$\begin{aligned}&r \cdot \big(\big(x_{i,j}(z)\big)_{i,j}\big) \cdot \begin{pmatrix} y_1 \ldots y_{m-1} & & & \\ & \ddots & & \\ & & y_1 & \\ & & & 1 \end{pmatrix} \\ &\quad = \left(\left(\prod_{k=i}^{j-1} r_{m-k}x_{i,j}\right)_{i,j}\right) \begin{pmatrix} r_1 \ldots r_{m-1}y_1 \ldots y_{m-1} & & & \\ & \ddots & & \\ & & r_1y_1 & \\ & & & 1 \end{pmatrix}.\end{aligned}$$

Multiplying the matrix on the right by a suitable element $k \in O(m, \mathbb{Z})$ with ± 1s on the diagonal will put it into Iwasawa form. Now to prove the lemma let $j = m$. Then, it follows that

$$x_{i,m}(\eta) = r_1 \dots r_{m-i} x_{i,m}(z),$$

and the lemma is proved. Note that the Iwasawa coordinates of η are just non-zero scalar multiples of the corresponding Iwasawa coordinates of z.

Combining Lemma 14 with Lemma 13, gives the following

$$\begin{aligned} u_n(\tau) &= u_n \left(r \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z \right) \\ &= \begin{pmatrix} r_1 \dots r_{n-1} & & & \\ & \ddots & & \\ & & r_1 r_2 & \\ & & & r_1 \end{pmatrix} \gamma u_n(z). \end{aligned} \tag{22}$$

Recall that the only τ coordinates that depend on the set $\{x_{i,n}$ where $1 \leqslant i \leqslant n-1\}$, i.e. on $u_n(z)$ are the coordinates of $u_n(\tau)$. The exact relationship is given in (22). Now in the calculation of $I_f(\mathbf{m}, y)$, if we integrate first with respect to $u(z)$, then the only factor in the integrand which depends on $u(z)$ is $e(-x_1(\tau))$.

Let P_n be the set of matrices in $GL(n)$ having a bottom row of $(0, \dots, 0, 1)$.

Lemma 15. *Let $\gamma \in GL(m-1, \mathbb{Z})$, $z \in \mathcal{H}^m$ and $u \in \mathbb{N}$. Let $\mathbf{r} = (r_1, \dots, r_{m-1}) \in (\mathbb{Z} \setminus 0)^{m-1}$ and let*

$$\tau \equiv r \begin{pmatrix} \gamma & 0 \\ 0 & 1 \end{pmatrix} z \mod \big(O(m, \mathbb{R}) Z(m, \mathbb{R})\big).$$

Then,

$$\int_0^1 \cdots \int_0^1 e(u x_1) e\big(-x_1(\tau)\big)\, dx_{1,m} \dots dx_{m-1,m} = \begin{cases} \delta_{r_1, \pm u} & \text{if } \gamma \in \pm P_{m-1} \text{ respectively}, \\ 0 & \text{else}. \end{cases}$$

Let $(a_1, \dots, a_{m-1})$ be the bottom row of γ, then we know the identity

$$x_1(\tau) = r_1 a_1 x_{1,m} + \dots + r_1 a_{m-1} x_{m-1,m}.$$

It follows that after integration in each $x_{i,m}$ where $1 \leqslant i < m-1$ the only nontrivial result occurs if $a_1 = \dots = a_{m-2} = 0$, because $r_1 \neq 0$. Since $\gamma \in GL(m-1, \mathbb{Z})$, it follows that $a_{m-1} = \pm 1$. So, $x_1(\tau) = \pm r_1 x_{m-1,m} = \pm r_1 x_1(z)$. Now, we can perform the integration in $x_{m-1,m}$ and the lemma is proved.

Since $r_1 \in \mathbb{N}$, using the above lemma once reduces the sum over γ in $\Gamma_\infty \setminus SL(n-1, \mathbb{Z})$ in our calculation of $I_f(\mathbf{m}, y)$ in line (19) to a sum over $\gamma \in \Gamma_\infty \setminus P(n-1, \mathbb{Z})$. But

$$\Gamma_\infty \setminus P(n-1, \mathbb{Z}) \equiv \Gamma_\infty \setminus SL(n-2, \mathbb{Z}).$$

As a result,

$$I_f(\mathbf{m}, y)$$

$$= \sum_{\substack{\gamma\in \\ \Gamma_\infty\backslash SL(n-2,\mathbb{Z})}} \sum_{\substack{\mathbf{r}=(r_2,\dots,r_{n-1}) \\ r_2,\dots,r_{n-1}\in\mathbb{Z}\backslash 0}} \overline{c_{m_1,\mathbf{r}}} \int_0^1 \cdots \int_0^1 \left[\prod_{i=2}^{n-1} e(m_i x_i) e\big(-x_i(\tau')\big) \overline{W}_J\big(y(\tau')\big)\right] \prod_{\substack{i,j=1 \\ i<j}}^{n-1} dx_{i,j},$$

where in the last step we have taken

$$\tau' \equiv \begin{pmatrix} r' & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & 0 & 0 \\ & 1 & 0 \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & 0 \\ 0 & 1 \end{pmatrix} \cdot \begin{pmatrix} m_1 y_1 \mathbb{I}_{n-1} & 0 \\ 0 & 1 \end{pmatrix},$$

where $z' \in \mathcal{H}^{n-1}$, r' is the matrix formed by the vector $(r_2, \dots, r_{n-1})$. It is easy to show that $x_i(\tau) = x_i(\tau')$ and $y_i(\tau) = y_i(\tau')$ because in the proofs of Lemmas 13 and 14 we showed that the only τ coordinates that depend on $u(z)$ are $u(\tau)$. We will just write τ instead of τ' from now.

Next we use the results of Lemmas 13 and 14 to put the matrix $r'\left(\begin{smallmatrix} \gamma & 0 \\ 0 & 1 \end{smallmatrix}\right)z'$ into Iwasawa form,

$$u_{n-1}(\tau) = \begin{pmatrix} r_2 \dots r_{n-1} & & & \\ & \ddots & & \\ & & r_2 r_3 & \\ & & & r_2 \end{pmatrix} \gamma u_{n-1}(z).$$

Now we can integrate over $u_{n-1}(z)$ using Lemma 15,

$$I_f(\mathbf{m}, y)$$

$$= \sum_{\substack{\gamma\in \\ \Gamma_\infty\backslash \pm SL(n-3,\mathbb{Z})}} \sum_{\substack{\mathbf{r}=(r_3,\dots,r_{n-1}) \\ r_3,\dots,r_{n-1}\in\mathbb{Z}\backslash 0}} \overline{c_{m_1,m_2,\mathbf{r}}} \int_0^1 \cdots \int_0^1 \prod_{i=3}^{n-1} e(m_i x_i) e\big(-x_i(\tau')\big) \overline{W}_J\big(y(\tau')\big) \prod_{\substack{i,j=1 \\ i<j}}^{n-2} dx_{i,j},$$

where in the last step we have taken

$$\tau \equiv \begin{pmatrix} r' & & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} \gamma & & & \\ & 1 & & \\ & & 1 & \\ & & & 1 \end{pmatrix} \cdot \begin{pmatrix} z' & & \\ & 1 & \\ & & 1 \end{pmatrix} \cdot \begin{pmatrix} m_1 m_2 y_2 y_1 \mathbb{I}_{n-2} & & \\ & m_1 y_1 & \\ & & 1 \end{pmatrix}.$$

The combination of the lemmas gives us the induction step that reduces the sum over γ to $\gamma = \Gamma_\infty \setminus \pm SL(2, \mathbb{Z})$. Applying the steps once again reduces γ to the identity. As a result, we end up with the following simplification, where we have let $du(x) = \prod_{i,j=1,\, i<j}^{2} dx_{i,j}$,

$$
\begin{aligned}
&I_f(\mathbf{m}, y) \\
&= \sum_{\substack{\gamma\in \\ \Gamma_\infty\backslash\pm SL(2,\mathbb{Z})}} \sum_{\substack{\mathbf{r}=(r_{n-2},r_{n-1}) \\ r_{n-2},r_{n-1}\in\mathbb{Z}\backslash 0}} \overline{c_{m_1,\ldots,m_{n-3},\mathbf{r}}} \int_0^1 \cdots \int_0^1 \prod_{i=n-2}^{n-1} e(m_i x_i) e\bigl(-x_i(\tau')\bigr) \overline{W}_J\bigl(y(\tau')\bigr)\, du(x) \\
&= \overline{c_{m_1,\ldots,m_{n-1}}}\, \overline{W}_J(my).
\end{aligned}
$$

Thus finishing the proof of Proposition 12 and also of the Main Theorem.

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